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On unbounded bodies with finite mass: asymptotic behaviour

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Abstract

There is introduced a class of barotropic equations of state (EOS) which become polytropic of index $n = 5$ at low pressure. One then studies asymptotically flat solutions of the static Einstein equations coupled to perfect fluids having such an EOS. It is shown that such solutions, in the same manner as the vacuum ones, are conformally smooth or analytic at infinity, when the EOS is smooth or analytic, respectively.

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In this Letter we introduce and study asymptotically flat solutions of the static Einstein equations for a source consisting of perfect-fluid matter with a specific class of equations of state (E'sOS). These E'sOS are chosen in such a way that there exist solutions where the support of the energy-momentum tensor is unbounded. In other words, there are solutions for which the fluid extends to infinity but becomes dilute at infinity fast enough so that the gravitational field is asymptotically flat, in particular the total mass is finite.

Before introducing the EOS we recall the field equations for a static metric on $\mathbf{R} \times M$ of the form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -e^{2U} dt^2 + e^{-2U} \gamma_{ij} dx^i dx^j, \tag{1}$$

where U, γ_{ij} depend only on the spatial coordinates x^i. For the source we have that

$$T_{\mu\nu} = \rho u_\mu u_\nu + p(g_{\mu\nu} + u_\mu u_\nu), \tag{2}$$

where $u_\mu u_\nu g^{\mu\nu} = -1$ and $u_\mu = f \xi_\mu$ with $\xi^\mu \partial_\mu = \partial / \partial t$. The Einstein equations

$$G_{\mu\nu} = 8\pi \, T_{\mu\nu} \tag{3}$$

give rise to

$$\Delta U = 4\pi(\rho + 3p) e^{-2U} \tag{4}$$

$$\mathcal{R}_{ij} = 2 D_i U D_j U - 16\pi \gamma_{ij} p e^{-2U}, \tag{5}$$

where Δ and $\mathcal{R}_{ij}$ are respectively the Laplacian and the Ricci tensor of γ_{ij}. From the contracted Bianchi identities for γ_{ij} we obtain the condition for hydrostatic equilibrium in the form

$$D_i p = -(\rho + p) D_i U. \tag{6}$$

If we have a barotropic EOS given by $\rho = \rho(p)$ and ρ and p are positive, both ρ and p can be viewed as functions of U. Then, from Eq. (6),

$$\frac{dp}{dU} = -(\rho + p). \tag{7}$$

Instead of prescribing $\rho(p)$ we can thus give an EOS in the parametric form

$$\rho = \rho(U), \qquad p = p(U). \tag{8}$$

A specific example, and the one which originally motivated the present work, is the so-called Buchdahl EOS [1] which can be written as

$$\rho = \rho_0 (1 - e^U)^5, \qquad p = \frac{1}{6} \rho_0 e^{-U} (1 - e^U)^6, \qquad 0 < \rho_0 = \text{const}, \tag{9}$$

which is equivalent to

$$p = \frac{1}{6} \frac{\rho^{6/5}}{(\rho_0 - \rho)^{1/5}}, \qquad 0 < \rho < \rho_0. \tag{10}$$

It is known [1] that, for each value of the central pressure, there exists a spherically symmetric solution of (4,5) which is asymptotically flat with matter extending to infinity ("Buchdahl solution"). In the paper [2] it was shown that any asymptotically flat solution of (4,5) with the EOS (10) coincides with a Buchdahl solution (in particular: is spherically symmetric). However in [2] it was assumed that, like in the vacuum case, asymptotic flatness at infinity implies conformal smoothness at infinity [3]. This gap was closed in the paper [11]. In the present work we consider the following, much more general, class of E'sOS:

$$(\rho + 3p)e^{-2U} = U^5 \phi(U^2) \tag{11}$$

$$pe^{-2U} = U^6 \psi(U^2). \tag{12}$$

Here $\psi : \mathbf{R}^+ \to \mathbf{R}$ is a smooth or analytic function and ϕ is chosen so that (7) is satisfied, i.e.

$$\phi(x) = -2[3\psi(x) + x\psi'(x)]. \tag{13}$$

One checks that (9) is a special case of (11, 12, 13) with ψ analytic. (Explicitly we have that $\psi(x) = \frac{\rho_0}{6}(\sinh\frac{\sqrt{x}}{2}/\frac{\sqrt{x}}{2})^6$.) In the case where the mass is positive we will have that U is strictly negative at least near infinity. Choosing, then, $\psi(0) > 0$, both ρ and p are positive and $\rho \sim 6(\psi(0))^{-1/6}p^{5/6}$ as $p \to 0$. Thus our E'sOS behave asymptotically like that of a polytrope of index 5.

In the following we assume that M is diffeomorphic to $\mathbf{R} \setminus \mathbf{B}_R(0)$ where $\mathbf{B}_R(0)$ is the closed ball of Euclidean radius R centered at the origin. The pair (U, γ_{ij}) is required to satisfy the decay conditions

$$U = O^\infty(1/r), \qquad \gamma_{ij} - \delta_{ij} = h_{ij} = O^\infty(1/r), \tag{14}$$

where $r^2 = x^i x^j \delta_{ij}$ and $O^\infty(F(r))$ means that the quantity in question is $O(F)$, its derivative is $O(F'(r))$, a.s.o. The Buchdahl solution satisfies these criteria. We will, in this paper, not study the general existence question of solutions satisfying the asymptotic conditions. The expectation is that, for any EOS subject to (11, 12, 13), there exist solutions having, say, finitely many but arbitrary multipole moments. (In the vacuum case this statement is also still a conjecture.) The situation with respect to global solutions, i.e. ones where $M \cong \mathbf{R}^3$ is different since one expects – and in many cases knows [4, 5] – that these have to be spherically symmetric and thus, for given EOS, are determined by a single parameter such as the central pressure. On the other hand, restricting to spherical symmetry from the outset and given an EOS satisfying (11, 12, 13) and such that ρ and p are positive: it is not clear whether solutions for some central pressure have finite radius, are not even asymptotically flat, or are at the borderline between these cases, namely of infinite radius but finite mass, i.e. have the asymptotic behaviour studied in this paper. (Strictly speaking, in the present context, we should look at the quantity U on the r.h. sides of (11, 12) as **some** parameter, not necessarily the gravitational potential, since, e.g. for finite radius, U does not go to zero at the surface of the body if it is required to vanish at infinity.)

Returning to the topic of this paper, namely asymptotic solutions, our main result is as follows:

Theorem: Let (U, γ_{ij}) be a solution of (4, 5) with EOS satisfying (11, 12, 13) on $\mathbf{R}^3 \setminus \mathbf{B}_R(0)$ with decay conditions (14) and positive mass. Then it is, in a sense in detail spelled out later, conformally smooth or analytic, depending on whether $\psi(x)$ is smooth or analytic.

The proof follows the pattern of that for the vacuum case [6, 3], so we shall merely outline it. We first derive an explicit expression for (U, γ_{ij}) near infinity up to order r^{-4}. To that effect we write (4, 5) as

$$\partial^2 U = h^{ij} U_{,ij} + \Gamma^i U_{,i} + 4\pi U^5 \phi(U^2) \tag{15}$$

with $\Gamma^i = \gamma^{k\ell}\Gamma^i_{k\ell}$ and $\partial^2 = \delta_{ij}\partial_i\partial_j$ and

$$\begin{aligned} \partial^2 h_{ij} \;&=\; 2\Lambda_{(i,j)} + h^{\ell m}(h_{ij,\ell m} + h_{\ell m,ij} - h_{kj,i\ell} - h_{i\ell,mj}) + \\ &\quad + 2\gamma^{mn}\gamma^{p\ell}(\Gamma_{nij}\Gamma_{mp\ell} - \Gamma_{ni\ell}\Gamma_{mpj}) - 4U_{,i}U_{,j} - \\ &\quad - 32\pi\gamma_{ij}U^6\psi(U^2), \end{aligned} \tag{16}$$

where $\gamma^{ij} = \delta_{ij} - h^{ij}$, $\Lambda_i = (h_{ij} - \frac{1}{2}\delta_{ij}h_{\ell\ell})_{ij}$, $h_{\ell\ell} = \delta_{i\ell}h_{i\ell}$. The r.h. side of (16) would be $O^\infty(r^{-4})$, if it were not for the term $2\Lambda_{(i,j)}$. But using a transformation of spatial coordinates of the form

$$\bar{x}^i = x^i + f^i(x) \tag{17}$$

one may choose f^i with $f^i(x) = O^\infty(\ln r)$ in such a way that, at least after suitably enlarging the radius R in $M = \mathbf{R}^3 \setminus \mathbf{B}_R(0)$, the quantity Λ_i satisfies

$$\Lambda_i = O^\infty((\ln^* r)r^{-3}), \tag{18}$$

where $\ln^* r$ denotes some power of $\ln r$. Thus we now have that

$$\partial^2 h_{ij} = O^\infty((\ln^* r)r^{-4}), \qquad \partial^2 U = O^\infty((l^* r)(r^{-4}). \tag{19}$$

Using standard (see e.g. [6]) estimates for the Poisson integral we infer

$$U \;=\; -\frac{M}{r} + O^\infty((\ln^* r)r^{-4}), \tag{20}$$

$$h_{ij} \;=\; \frac{t_{ij}}{r} + O^\infty((\ln^* r)r^{-4}), \tag{21}$$

where M (the mass) and $t_{ij} = t_{(ij)}$ are constants. Using the gauge condition (18) it now follows that t_{ij} has to vanish. We next insert (20, 21) into the r.h. sides of (15, 16) and refine the gauge so that

$$\Lambda_i = O^\infty((\ln^* r)r^{-4}) \tag{22}$$

to obtain

$$U = -\frac{M}{r} + \frac{M_i x^i}{r^3} + O^\infty((\ln^* r)r^{-3}), \tag{23}$$

$$h_{ij} = M^2\left(\frac{x_i x_j}{r^4} - -\frac{\delta_{ij}}{r^2}\right) + \frac{t_{ijk}x^k}{r^3} + O^\infty((\ln^* r)r^{-3}), \tag{24}$$

where $x_i = \delta_{ij}x^j$ and $t_{ijk} = t_{(ij)k}$ are constants with $t_{iij} = 0$.

As shown in [7] the constants t_{ijk} can be gauged away in a manner respecting the condition (22). The constant M_i in (23) can also be disposed of by the translation $\bar{x}^i = x^i - M_i/M$ (using that M is positive, in particular nonzero). Inserting the expansions (23, 24) into the r.h. sides of (15, 16), the above pattern repeats itself. Thus we obtain

$$\begin{aligned} U &= -\frac{M}{r} - \frac{M^3(3cM^2-4)}{12r^3} + \frac{M_{ij}x^ix^j}{r^5} + \\ &\quad + \frac{M_{ijk}x^ix^jx^k}{r^7} + O^\infty((\ln^* r)r^{-5}), \end{aligned} \tag{25}$$

$$\begin{aligned} h_{ij} &= \frac{M^2x_ix_j}{r^4} - \frac{M^2\delta_{ij}}{r^2} + \frac{2M^4x_ix_j}{3r^6} - \frac{2M^4\delta_{ij}}{9r^4} - \frac{cM^6x_ix_j}{r^6} - \\ &\quad + \frac{5MM_{k\ell}x^kx^\ell x_ix_j}{3r^6} + \frac{4Mx_{(i}M_{j)k}x^k}{r^6} - \frac{28MM_{ij}}{27r^4} - \\ &\quad - \frac{5MM_{k\ell}x^kx^\ell x_ix_j}{r^8} + O^\infty((\ln^* r)r^{-5}), \end{aligned} \tag{26}$$

where $c = -16\pi\psi(0)$ and the constants M_{ijk} – the octopole moment – satisfy $M_{ijk} = M_{(ijk)}$ and $M_{iij} = 0$.

Next there comes a crucial observation. Let

$$\omega = \frac{U^2}{M^2} \tag{27}$$

and define "unphysical" coordinates by

$$\widetilde{x}^i = \frac{x^i}{r^2}. \tag{28}$$

It is then not difficult to check that ω, as a function of $\widetilde{x}^i$, is in the open ball $\mathbf{B}_{1/R}(0) = \widetilde{M}$, four times continuously differentiable, in fact the fourth derivatives are Hölder continuous with index α $(0 < \alpha < 1)$, in short: $\omega \in C^{4,\alpha}$. (Note that $\omega|_\Lambda = 0$, $D_i\omega|_\Lambda = 0$, where Λ is the point-at-infinity, i.e. $\widetilde{x}^i = 0$.) Furthermore $\widetilde{\gamma}_{ij}$ defined by

$$\widetilde{\gamma}_{ij} = \omega^2\gamma_{ij} \tag{29}$$

is also $C^{4,\alpha}$ in the coordinate system given by $\widetilde{x}^i$. Armed with this information we can now try to derive field equations for $(\omega, \widetilde{\gamma}_{ij})$ on $\widetilde{M}$. We use standard identities on conformal rescalings and follow the pattern of [3]. Writing, for simplicity, again γ_{ij} for the unphysical metric $\widetilde{\gamma}_{ij}$ and setting $M = 1$ without loss we obtain the equations

$$\Delta\omega = 3\mathcal{R} + \alpha(\omega), \tag{30}$$

$$\omega\mathcal{R}_{ij} = \frac{1}{2}(D_i\omega)(D_j\omega) - D_iD_j\omega + \frac{1}{3}\gamma_{ij}\Delta\omega + \gamma_{ij}\omega\beta(\omega), \tag{31}$$

where

$$\frac{1}{8\pi}\alpha(\omega) = \omega\phi(\omega) + 18\omega\psi(\omega) + 12\phi(\omega), \quad (32)$$

$$\frac{1}{8\pi}\beta(\omega) = -2\omega\psi(\omega) - \frac{4}{3}\phi(\omega). \quad (33)$$

Contracting (31) and taking the gradient, the vacuum terms without prefactor ω cancel and we obtain

$$D_i\mathcal{R} = -(D^j\omega)\mathcal{R}_{ij} + (D_i\omega)(\mathcal{R} - 2\beta + 3\beta') + (D_i\omega)\omega^{-1}\left(\frac{\alpha}{3} + 3\beta\right). \quad (34)$$

But, luckily,

$$\frac{\alpha}{3} + 3\beta = \frac{1}{3}\omega\phi, \quad (35)$$

so the dangerous last term in (34) is also regular. A similar miracle occurs after taking D_k of (31) and antisymmetrizing with respect to j and k. Using the Ricci identity, the relation

$$\mathcal{R}_{ijk\ell} = 2(\gamma_{i[k}\mathcal{R}_{\ell]j} - \gamma_{j[k}\mathcal{R}_{i]\ell} - \gamma_{i[k}\gamma_{j]\ell}), \quad (36)$$

valid in 3 dimensions and, again, (30, 31) and (34), we finally arrive at

$$\omega\left[D_{[k}\mathcal{R}_{j]i} - \frac{1}{2}(D_{[k}\omega)\mathcal{R}_{j]i}\right] = \\ = (D_{[k}\omega)\gamma_{j]i}\left[-\frac{\alpha}{6} - \frac{\omega\beta}{2} - 2\beta + 3\beta' + \omega^{-1}\left(\frac{\alpha}{3} + 3\beta\right) + (\omega\beta)' + \frac{\alpha'}{3}\right]. \quad (37)$$

Again, by virtue of (32, 33), the dangerous terms drop out. Thus there results an expression

$$D_k\mathcal{R}_{ij} = D_j\mathcal{R}_{ki} + (D_{[k}\omega)\mathcal{R}_{j]i} + (D_{[k}\omega)\gamma_{j]i}\varepsilon, \quad (38)$$

where ε is a known smooth or analytic function of ω. Now take D^k of (38). Using the Ricci and Bianchi identity together with Eq. (36), and (30, 31) to eliminate second derivatives of ω, we obtain an equation for $\Delta\mathcal{R}_{ij}$, with a regular r.h. side depending merely on γ_{ij}, $\mathcal{R}_{ij}$, ω and $D_i\omega$. Thus, writing

$$\Delta\omega = 3\mathcal{R} + \alpha \quad (39)$$

$$\mathcal{R}_{ij} = \sigma_{ij} \quad (40)$$

$$\Delta\sigma_{ij} = \ldots \quad (41)$$

and transforming to harmonic coordinates, the set of equations (39, 40, 41) becomes an elliptic system for $(\omega, \gamma_{ij}, \sigma_{ij})$ with smooth or analytic coefficients. Consequently $(\omega, \gamma_{ij}, \sigma_{ij})$ which by our previous analysis have been $C^{2,\alpha}$, are actually smooth or analytic, by virtue of a famous theorem of Morrey (see Thm 6.7.6 of [8]), and the proof is complete. Let us finally point out that the solutions discussed in the present work can be characterized by multipole moments [9] in completely the same way as the vacuum solutions [3]. The details can be found in the diploma thesis by one of us (M.K.) [10].

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